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EXTRACTING SECOND–ORDER STRUCTURES FROM SINGLE–INPUT STATE–SPACE MODELS: APPLICATION TO MODEL ORDER REDUCTION

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This paper focuses on the model order reduction problem of second-order form models. The aim is to provide a reduction procedure which guarantees the preservation of the physical structural conditions of second-order form models. To solve this problem, a new approach has been developed to transform a second-order form model from a state-space realization which ensures the preservation of the structural conditions. This new approach is designed for controllable single-input state-space realizations with real matrices and has been applied to reduce a single-input second-order form model by balanced truncation and modal truncation.

Keywords: second-order form model, preservation of the structural conditions, balanced truncation, modal truncation.

Notation

X^T is the transpose of the matrix X.
$\bar{X}$ and $|X|$ denote respectively the conjugate and the modulus of the complex matrix X.
$X > 0$ (resp. $X \geq 0$) is a positive definite (resp. semi-definite) matrix.
$X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ is a diagonal matrix with entries $x_1, x_2, \ldots, x_n$.
$\lambda_i(X)$ is the i-th eigenvalues of the matrix X.
$\mathrm{Re}(z)$ is the real part of the complex number z.
$\mathbb{O}$ and $\mathbb{I}$ are respectively the zero and the identity matrix with adequate dimensions.

1. Introduction

The main purpose of Model Order Reduction (MOR) is to reduce the complexity of a model while preserving its behaviour as much as possible, usually according to an approximation error (Schilders, 2008). Depending on the research domain, MOR seeks different goals. In control theory, the goals of MOR are to save computational simulation costs and/or obtain simplified control laws. Therefore, only the behaviour of the system is preserved, and, generally, the specific structure defined by the physical system is lost. In other research domains such as electric circuit design, mechanical system design, fluid dynamics, thermodynamical processes or structural analysis, the goal of MOR is to simplify the model description. Therefore, the structure of the system must be preserved. In these domains, a particular class of structured models describes systems with a structure defined by the physical laws: Second-Order Form Models (SOFMs). Parameters of these models are generalized mass, damping and stiffness which can be linked to the parameters of mechanical, electrical, fluid or thermodynamical systems (Dorf and Bishop, 2008, Chapter 2). If a system is described by several differential equations, SOFMs are represented in a matrix form. In this case, generalized mass, damping and stiffness matrices must satisfy the *structural conditions*.

In control theory, the reduction procedures are generally based on the well-known moment matching, Krylov's subspace, the singular value or the eigenvalue (see, e.g., Antoulas, 2005; Ersal *et al.*, 2007; Fortuna *et al.*, 1992; Li and White, 2001). These methods are efficient in terms of the approximation error of the reduced model. The main drawback is the difficulty to find a physical system corresponding to the reduced model.

Contrary to the control theory approach, the reduction procedures used in the structural analysis approaches ensure the physical feasibility of the reduced model. For instance, the Guyan reduction, dynamic reduction or improved reduced systems are methods which preserve the second-order form and the *structural condi-*

tions (Koutsovasilis and Beitelschmidt, 2008). However, these methods are generally less efficient in terms of the approximation error of the reduced model.

For two decades, studies in control theory have adapted MOR procedures for structured systems and, in particular, for SOFMs. The main goal is to reduce the model with an efficient approximation error while preserving the second-order form. Some use structure preservation techniques to reduce a model (Bai *et al*., 2008; Li and Bai, 2006), others deal directly with SOFMs (Freund, 2005; Salimbahrami, 2005). An interesting technique of MOR based on singular values is the well-known balanced truncation. This method ensures the preservation of stability, controllability and observability properties in the reduced model. Moreover, upper and lower bounds of the approximation error are given. A first adaptation of balanced truncation for the SOFM was proposed by Meyer and Sirnivasan (1996). Further, Chahlaoui *et al*. (2006) improved the method with the SOBT (Second-Order Balanced Truncation) algorithm.

Between classical balanced truncation and SOBT, differences remain in Gramians. SOBT methods are based on the definition of two pairs of second-order Gramians, called *position* and *velocity* Gramians. Stykel (2006) as well as Reis and Stykel (2007) proposed methods to balance models according to one or both of the Gramians pairs, namely, SOBTp and SOBTpv for position and position-velocity, respectively. If the adaptation differs, according to the authors, three remarks can be made: First, the approximation error of the reduced model is generally greater than the approximation error of the model reduced through classic balanced truncation. Secondly, the bound of the approximation error cannot be computed yet. Thirdly, the *structural conditions* are not necessarily preserved.

The aim of this paper is to propose a new method to reduce an SOFM. This method is designed for controllable single-input models with real parameters and helps to preserve the *structural conditions* as well as the properties and the approximation error of the balanced truncation.

Section 2 present SOFMs, *structural conditions* and reduction framework. Section 3 describes a new method to transform a single-input model into an SOFM. Section 4 presents balanced truncation and modal truncation for SOFM reduction with the preservation of the *structural conditions*. Based on two examples of the SLICOT Benchmark[1], Section 5 gives numerical results preceding the conclusion.

2. Problem presentation

Several mathematical formulations have been developed to model mechanical systems. A common representation is the state-space one due to its simplicity of manipulation. But, in the reduction procedure, the physical interpretation of the model is generally lost. To keep this physical interpretation after the reduction step, the SOFM formulation of Linear Time Invariant (LTI) systems is considered. The general formulation of SOFM is given by

$$\Sigma_{\text{sofm}}: \quad \begin{cases} \mathcal{M}\ddot{q} + \mathcal{C}\dot{q} + \mathcal{K}q = Fu, \\ y = G_1 q + G_2\dot{q} + G_3\ddot{q}, \end{cases} \tag{1}$$

with

$$q \in \mathbb{R}^{n_q\times 1}, \qquad \mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbb{R}^{n_q\times n_q},$$
$$F \in \mathbb{R}^{n_q\times m}, \qquad G_1, G_2, G_3 \in \mathbb{R}^{p\times n_q},$$

where $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are respectively the mass, damping and stiffness matrices of the system, q is the vector of the coordinates with dimension n_q, m is the number of inputs and p the number of outputs. To ensure the physical interpretation and the stability of the SOFM, the *structural conditions* must be respected (Meyer and Sirnivasan, 1996):

$$\begin{cases} \mathcal{M} = \mathcal{M}^T > 0, \\ \mathcal{K} = \mathcal{K}^T \geq 0, \\ \mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 \text{ with } \mathcal{C}_1 = \mathcal{C}_1^T \geq 0,\ \mathcal{C}_2 = -\mathcal{C}_2^T. \end{cases} \tag{2}$$

The structural condition for the mass matrix comes from the system kinematic energy, given by $E_k = \frac{1}{2}\dot{q}^T\mathcal{M}\dot{q}$. It can be shown that $\mathcal{M}$ is symmetric and positive definite (all coordinates must have inertia). For the same reason, the study of the potential energy given by $E_p = \frac{1}{2}\dot{q}^T\mathcal{K}\dot{q}$ implies that $\mathcal{K}$ is symmetric and positive semi-definite (possibility of a “dampingless” coordinate). Gyroscopic forces $f_{\mathcal{C}_2} = -\mathcal{C}_2 q$ arise when rotors are present or when q is defined in a rotative frame. Dissipative forces $f_{\mathcal{C}_1} = -\mathcal{C}_1 q$ never add energy to the system, and therefore $\mathcal{C}_1$ is positive semi-definite (Hughes and Skelton, 1980). Finally, the symmetry of matrices can be obtained by action-reaction principle between coordinates.

In this study, G_3 and $\mathcal{C}_2$ are assumed to be zero and the system is single-input, i.e., F is an n_q-dimensional vector. Since $\mathcal{M}$ is positive definite, $\mathcal{M}$ is invertible.

The aim of the reduction is to find a new SOFM:

$$\hat{\Sigma}_{\text{sofm}}: \quad \begin{cases} \hat{\mathcal{M}}\ddot{\hat{q}} + \hat{\mathcal{C}}\dot{\hat{q}} + \hat{\mathcal{K}}\hat{q} = \hat{F}u, \\ \hat{y} = \hat{G}_1\hat{q} + \hat{G}_2\dot{\hat{q}}, \end{cases} \tag{3}$$

with

$$\hat{q} \in \mathbb{R}^{\hat{n}_q\times 1}, \qquad \hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{K}} \in \mathbb{R}^{\hat{n}_q\times \hat{n}_q},$$
$$\hat{F} \in \mathbb{R}^{\hat{n}_q\times 1}, \qquad \hat{G}_1, \hat{G}_2 \in \mathbb{R}^{p\times \hat{n}_q},$$

where we have $\hat{n}_q < n_q$, $\hat{\mathcal{M}} = \hat{\mathcal{M}}^T > 0$, $\hat{\mathcal{C}} = \hat{\mathcal{C}}^T \geq 0$, $\hat{\mathcal{K}} = \hat{\mathcal{K}}^T \geq 0$, and such that the following properties are satisfied (Gugercin, 2004):

[1]Available at `www.icm.tu-bs.de/NICONET/index.html`.

1. The approximation error $||y - \hat{y}||$ is small, and there exists a global error bound.

2. System properties (stability, passivity, structure, etc.) are preserved.

3. The procedure is computationally efficient.

In this paper, the approximation error is evaluated using the $\mathcal{H}_\infty$-norm of the relative error model.

The system (1) can be written in the following state-space realization $\Sigma_{\text{ss}} = [\begin{smallmatrix} A & B \\ C & \mathbb{O} \end{smallmatrix}]$:

$$\Sigma_{\text{ss}} : \quad \begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{4}$$

with

$$A = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}\mathcal{C} \end{pmatrix} \in \mathbb{R}^{2n_q \times 2n_q},$$
$$B = \begin{pmatrix} \mathbb{O} \\ \mathcal{M}^{-1}F \end{pmatrix} \in \mathbb{R}^{2n_q \times 1},$$
$$C = \begin{pmatrix} G_1 & G_2 \end{pmatrix} \in \mathbb{R}^{p \times 2n_q}.$$

The reduced SOFM is also rewritten in the state-space realization $\hat{\Sigma}_{\text{ss}} = \left[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \mathbb{O} \end{smallmatrix}\right]$ such that

$$\hat{\Sigma}_{\text{ss}} : \quad \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \\ \hat{y} = \hat{C}\hat{x}, \end{cases} \tag{5}$$

with

$$\hat{A} \in \mathbb{R}^{2\hat{n}_q \times 2\hat{n}_q}, \quad \hat{B} \in \mathbb{R}^{2\hat{n}_q \times 1}, \quad \hat{C} \in \mathbb{R}^{p \times 2\hat{n}_q}.$$

To have the same approximation error as for the first-order model reduction, the reduction procedure is based on the state-space realization (4) of the SOFM. From the reduced state-space realization (5), the proposed solution consists in deducing an SOFM which preserves the *structural conditions*. The different steps of the process are summarized by the following diagram:

$$\Sigma_{\text{sofm}} \xrightarrow{\text{Equation (4)}} \Sigma_{\text{ss}} \xrightarrow{\text{Section 4}} \hat{\Sigma}_{\text{ss}} \xrightarrow{\text{Section 3}} \hat{\Sigma}_{\text{sofm}}.$$

3. Second-order form reconstruction from a single-input state-space realization

The transformation of an SOFM into an state-space realization can be easily performed (see Eqn. (4)) but the inverse transformation requires more attention. Several methods have been presented (Friswell, 1999; Houlston, 2006; Salimbahrami, 2005), but none of these preserve the *structural conditions*. To the authors' knowledge, the first method to transform a state-space realization into a second-order form model was proposed by Meyer and Sirnivasan (1996). In this paper, it is shown that for all minimal single-input state-space realizations there exists a second-order form realization. If A has distinct eigenvalues, the second-order form realization may be constructed such that both C and K are diagonals. According to the authors, the proposed method is not numerically attractive.

In this section, a new approach to find an SOFM from some single-input state-space realization is proposed. The approach ensures the preservation of the *structural conditions* if the state-space is stable and controllable. As in the work of Meyer and Sirnivasan (1996), the diagonalization of A must be achieved, and therefore A is assumed to be diagonalizable, which is the case for most physical systems. However, there exist particular systems for which diagonalization cannot be performed, e.g., when critical damping occurs (Tisseur and Meerbergen, 2001; Gohberg *et al.*, 1982). A sufficient condition to ensure A diagonalization is that A must have $2n_q$ distinct eingenvalues.

The proposed method is presented in four steps:

1. diagonalization of the state matrix A,

2. computation of the second-order form,

3. guarantee of the realness of the matrices,

4. extraction of the SOFM from the new state-space realization.

3.1. First step: Diagonalization of the state matrix A.

The first step expresses a state-space realization in its modal basis. Therefore a state-space realization $\left[\begin{smallmatrix} A & B \\ C & \mathbb{O} \end{smallmatrix}\right]$ becomes a new state-space $\left[\begin{smallmatrix} A_d & B_d \\ C_d & \mathbb{O} \end{smallmatrix}\right]$ where the state matrix A_d is diagonal (assuming that A has $2n_q$ distinct eingenvalues).

Consider the eigenvalue decomposition of $A \in \mathbb{R}^{2n_q \times 2n_q}$. Due to the realness of A, the eigenvalues are real or come in n_c complex conjugate pairs. We can order them such that

$$\Phi^{-1}A\Phi = A_d = \begin{pmatrix} \Lambda_1 & \mathbb{O} \\ \mathbb{O} & \Lambda_2 \end{pmatrix}, \tag{6}$$
$$\Lambda_1 = \begin{pmatrix} \Lambda_c & \mathbb{O} \\ \mathbb{O} & \Lambda_{r1} \end{pmatrix} \in \mathbb{C}^{n_q \times n_q},$$
$$\Lambda_2 = \begin{pmatrix} \bar{\Lambda}_c & \mathbb{O} \\ \mathbb{O} & \Lambda_{r2} \end{pmatrix} \in \mathbb{C}^{n_q \times n_q},$$

where

- $\Lambda_c \in \mathbb{C}^{n_c \times n_c}$ and $\bar{\Lambda}_c \in \mathbb{C}^{n_c \times n_c}$ are diagonal matrices of the complex eigenvalues,

- $\Lambda_{r1} \in \mathbb{R}^{(n_q - n_c) \times (n_q - n_c)}$ and $\Lambda_{r2} \in \mathbb{R}^{(n_q - n_c) \times (n_q - n_c)}$ are two diagonal matrices of the real eigenvalues.

With block partition of the matrices Φ et Φ^{-1} such that

$$\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} \Phi i_1 \\ \Phi i_2 \end{pmatrix}, \tag{7}$$

with $\Phi_1, \Phi_2 \in \mathbb{C}^{2n_q \times n_q}$ and $\Phi i_1, \Phi i_2 \in \mathbb{C}^{n_q \times 2n_q}$, matrices B_d and C_d are obtained by

$$B_d = \begin{pmatrix} \Phi i_1 \\ \Phi i_2 \end{pmatrix} B, \qquad C_d = C \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix}.$$

3.2. Second step: Computation of the second-order form. Since Eqn. (4) is a state-space realization of an SOFM, the transformation must establish the appropriate location of the zero and the identity matrix into A_d and B_d. A first solution was proposed by Friswell (1999) for an SOFM without velocity and acceleration observation matrices ($G_2 = G_3 = \mathbb{O}$). Based on the work of Prells and Lancaster (2005) about *Structural Preserving Equivalence* (SPE) transformation for vibrating systems, Houlston (2006) proposed the following transformation matrix:

$$T = \begin{pmatrix} X \\ XA_d \end{pmatrix}^{-1}, \tag{8}$$

with $X \in \mathbb{R}^{n_q \times n}$ being a full rank matrix. Noting that

$$XA_d \begin{pmatrix} X \\ XA_d \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \end{pmatrix}, \tag{9}$$

T transforms the state matrix A_d into a state-space realization satisfying Eqn. (4):

$$\begin{aligned} A_T = T^{-1} A_d T &= \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ A_{T1} & A_{T2} \end{pmatrix}, \\ B_T = T^{-1} B_d &= \begin{pmatrix} B_{T1} \\ B_{T2} \end{pmatrix}, \\ C_T &= C_d T, \end{aligned} \tag{10}$$

The condition that $B_{T1} = \mathbb{O}$ helps to determine the matrix X. According to (4), B_{T1} must be equal to zero. Therefore, considering the block partition of T^{-1}, X must satisfy

$$XB_d = \mathbb{O}. \tag{11}$$

Friswell (1999), Meyer and Sirnivasan (1996) as well as Salimbahrami (2005) seek to find X respecting (11) directly. Here, a block partition of X into two matrices X_1 and X_2 such that $X = \begin{pmatrix} X_1^{-1} & X_2^{-1} \end{pmatrix}$ gives

$$\begin{aligned} \begin{pmatrix} X_1^{-1} & X_2^{-1} \end{pmatrix} \begin{pmatrix} \Phi i_1 B \\ \Phi i_2 B \end{pmatrix} &= \mathbb{O}, \\ X_1^{-1} \Phi i_1 B &= -X_2^{-1} \Phi i_2 B. \end{aligned} \tag{12}$$

In the SIMO case, $\Phi i_1 B$ and $\Phi i_2 B$ are vectors. Consequently, the solution to Eqn. (12) is not unique. Among all the solutions, if the model is controllable, setting

$$\begin{aligned} X_1 &= -\operatorname{diag}\begin{pmatrix} b_{d1}, & b_{d2}, & \ldots, & b_{dn_q} \end{pmatrix}, \\ X_2 &= \operatorname{diag}\begin{pmatrix} b_{dn_q+1}, & b_{dn_q+2}, & \ldots, & b_{dn} \end{pmatrix}, \end{aligned} \tag{13}$$

where b_{di} is the i-th component of vector B_d, allows finding a solution where X_1 and X_2 are directly constructed from B_d without computation.

From (13), it is clear that

$$\begin{cases} X_1^{-1} \Phi i_1 B &= -\mathbb{1}_{nq \times 1}, \\ X_2^{-1} \Phi i_2 B &= \mathbb{1}_{nq \times 1}, \end{cases} \tag{14}$$

where $\mathbb{1}_{nq \times 1}$ is an n_q column vector with all entries equal to 1.

To show the existence of X_1^{-1} and X_2^{-1}, examine the state-space realization $\left[\begin{smallmatrix} A_d & B_d \\ C_d & \mathbb{O} \end{smallmatrix}\right]$. In a modal basis, the state-space realization represents a set of several independent differential equations. In the SIMO case, since A_d is a diagonal matrix of a controllable model, all differential equations are controllable. This implies that the vector $\Phi^{-1} B$ has non-zero entries.

Finally, as X_1, X_2, Λ_1 and Λ_2 are diagonal,

$$T^{-1} = \begin{pmatrix} X_1^{-1} & X_2^{-1} \\ X_1^{-1}\Lambda_1 & X_2^{-1}\Lambda_2 \end{pmatrix},$$

$$T = \begin{pmatrix} X_1 \Lambda_2 & -X_1 \\ -X_2 \Lambda_1 & X_2 \end{pmatrix} \cdot \begin{pmatrix} (\Lambda_2 - \Lambda_1)^{-1} & \mathbb{O} \\ \mathbb{O} & (\Lambda_2 - \Lambda_1)^{-1} \end{pmatrix}.$$

Therefore, T transforms the state-space realization $\left[\begin{smallmatrix} A_d & B_d \\ C_d & \mathbb{O} \end{smallmatrix}\right]$ into a new state-space realization $\left[\begin{smallmatrix} A_T & B_T \\ C_T & \mathbb{O} \end{smallmatrix}\right]$:

- *Matrix A_T,*

$$\begin{aligned} A_T = T^{-1} A_d T &= \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\Lambda_1 \Lambda_2 & \Lambda_2 + \Lambda_1 \end{pmatrix} \\ &\triangleq \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ A_{T1} & A_{T2} \end{pmatrix}, \end{aligned} \tag{15}$$

where A_{T1} and A_{T2} are diagonal,

$$A_{T1} = -\Lambda_1 \Lambda_2 = -\begin{pmatrix} |\Lambda_c|^2 & \mathbb{O} \\ \mathbb{O} & \Lambda_{r1}\Lambda_{r2} \end{pmatrix},$$

$$A_{T2} = \Lambda_1 + \Lambda_2 = \begin{pmatrix} \Lambda_c + \bar{\Lambda}_c & \mathbb{O} \\ \mathbb{O} & \Lambda_{r1} + \Lambda_{r2} \end{pmatrix}.$$

Moreover, A_{T1} and A_{T2} have negative entries due to the stability condition.

• *Matrix* B_T,

$$B_T = T^{-1}B_d = \begin{pmatrix} X_1^{-1}\Phi i_1 + X_2^{-1}\Phi i_2 \\ X_1^{-1}\Lambda_1\Phi i_1 + X_2^{-1}\Lambda_2\Phi i_2 \end{pmatrix} B. \tag{16}$$

Since X_1 and X_2 are defined such that $X_1^{-1}\Phi i_1 B = -\mathbb{1}_{nq\times 1}$ and $X_2^{-1}\Phi i_2 B = \mathbb{1}_{nq\times 1}$, we get

$$B_T = \begin{pmatrix} \Lambda_2 - \Lambda_1 & \mathbb{O} \\ \mathbb{O} & \Lambda_2 - \Lambda_1 \end{pmatrix} \begin{pmatrix} \mathbb{O} \\ \mathbb{1}_{nq\times 1} \end{pmatrix}. \tag{17}$$

Clearly, the entries of B_T are real or purely imaginary numbers.

• *Matrix* C_T,

$$C_T = C \begin{pmatrix} \Phi_1 X_1 \Lambda_2 - \Phi_2 X_2 \Lambda_1 & X_1\Lambda_2 - X_2\Lambda_1 \end{pmatrix} \begin{pmatrix} (\Lambda_2-\Lambda_1)^{-1} & \mathbb{O} \\ \mathbb{O} & (\Lambda_2-\Lambda_1)^{-1} \end{pmatrix}. \tag{18}$$

Finally, A_T, B_T and C_T have the required structure, but their realness is not yet guaranteed. This is the aim of the next section.

3.3. Third step: Guarantee of the realness of the matrices. By examining Eqn. (17), it can be noticed that complex entries of B_T are provided by $(\Lambda_1 - \Lambda_2)$ (due to the structure of Λ_1 and Λ_2 from Eqn. (6)).

To have real entries in B_T, the transformation matrix U^{-1} applied to B_T must eliminate $(\Lambda_1 - \Lambda_2)$:

$$U = \begin{pmatrix} \Lambda_2 - \Lambda_1 & \mathbb{O} \\ \mathbb{O} & \Lambda_2 - \Lambda_1 \end{pmatrix}.$$

The transformation U transforms the state-space realization $\left[\begin{smallmatrix} A_T & B_T \\ C_T & \mathbb{O} \end{smallmatrix}\right]$ into a new state-space realization $\left[\begin{smallmatrix} A_f & B_f \\ C_f & \mathbb{O} \end{smallmatrix}\right]$:

• *Matrix* A_f,

$$A_f = U^{-1}A_T U = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\Lambda_2\Lambda_1 & \Lambda_2 + \Lambda_1 \end{pmatrix}. \tag{19}$$

Since Λ_2 and Λ_1 are diagonal, A_T remains unchanged.

• *Matrix* B_f,

$$B_f = U^{-1}B_T = \begin{pmatrix} \mathbb{O} \\ \mathbb{1}_{nq\times 1} \end{pmatrix}. \tag{20}$$

• *Matrix* C_f,

$$\begin{aligned} C_f &= C_T U \\ &= C \begin{pmatrix} \Phi_1 X_1 \Lambda_2 - \Phi_2 X_2 \Lambda_1 & -\Phi_1 X_1 + \Phi_2 X_2 \end{pmatrix}. \end{aligned} \tag{21}$$

In order to prove the realness of C_f, the block partitioning of Φ_1, Φ_2, Φi_1 and Φi_2 into real and complex parts yields

$$\Phi_1 = \begin{pmatrix} \Phi_c & \Phi_{r1} \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \bar{\Phi}_c & \Phi_{r2} \end{pmatrix},$$

where $\Phi_c, \bar{\Phi}_c \in \mathbb{C}^{n_q\times n_c}$, and $\Phi_{r1}, \Phi_{r2} \in \mathbb{R}^{n_q\times(n_q-n_c)}$.

$$\Phi i_1 = \begin{pmatrix} \Phi i_c \\ \Phi i_{r1} \end{pmatrix}, \quad \Phi i_2 = \begin{pmatrix} \bar{\Phi} i_c \\ \Phi i_{r2} \end{pmatrix},$$

where $\Phi i_c, \bar{\Phi} i_c \in \mathbb{C}^{n_c\times n_q}$, and $\Phi i_{r1}, \Phi i_{r2} \in \mathbb{R}^{(n_q-n_c)\times n_q}$.

Index c refers to the complex part and index r refers to the real part. Since the entries of the diagonal matrices X_1 and X_2 are respectively the entries of the two following column vectors $-\Phi_{i1}B$ and $\Phi_{i2}B$, the first n_c rows of $-\Phi_1 X_1$ are complex conjugates of first n_c rows of $\Phi_2 X_2$ and the last $n_q - n_c$ rows of $-\Phi_1 X_1$ and of $\Phi_2 X_2$ are real. Hence, $-\Phi_1 X_1 + \Phi_2 X_2$ is a matrix with real entries.

Complex and real block partitioning of Λ_1 and Λ_2 yields

$$\begin{aligned} \Lambda_1 &= \begin{pmatrix} \Lambda_c & \mathbb{O} \\ \mathbb{O} & \Lambda_{r1} \end{pmatrix} \in \mathbb{R}^{n_q\times n_q}, \\ \Lambda_2 &= \begin{pmatrix} \bar{\Lambda}_c & \mathbb{O} \\ \mathbb{O} & \Lambda_{r2} \end{pmatrix} \in \mathbb{R}^{n_q\times n_q}. \end{aligned} \tag{22}$$

Therefore the first n_c rows of $\Phi_1 X_1 \Lambda_2$ are the complex conjugates of the first n_c rows of $-\Phi_2 X_2 \Lambda_1$ and the last $n_q - n_c$ rows of $\Phi_1 X_1 \Lambda_2$ and $-\Phi_2 X_2 \Lambda_1$ are real. Hence, $\Phi_1 X_1 \Lambda_2 - \Phi_2 X_2 \Lambda_1$ is a matrix with real entries. Thus C_f is a matrix with real entries.

3.4. Fourth step: Extraction of the second-order form matrices. This last step consists in the extraction of $\mathcal{M}$, $\mathcal{C}$, $\mathcal{K}$, F, G_1 and G_2 of the second-order form model from the state-space realization $\left[\begin{smallmatrix} A_f & B_f \\ C_f & \mathbb{O} \end{smallmatrix}\right]$.

With no loss of generality, assuming $\mathcal{M} = \mathbb{I}$ to normalize the SOFM gives

$$\begin{cases} \mathcal{M} = \mathbb{I}, \\ \mathcal{C} = -\Lambda_1 - \Lambda_2, \\ \mathcal{K} = \Lambda_1\Lambda_2, \\ F = \mathbb{1}_{nq\times 1}, \\ G_1 = C\left(\Phi_1 X_1 \Lambda_2 - \Phi_2 X_2 \Lambda_1\right), \\ G_2 = C\left(-\Phi_1 X_1 + \Phi_2 X_2\right). \end{cases} \tag{23}$$

For a single-input, stable, controllable state-space realization of an even dimension, with real matrices and A diagonalizable, an SOFM can be determined. The stability condition ensures that $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ will be positive definite. The realness of A ensures that $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ will be real. Moreover, $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ are diagonal. Consequently, $\mathcal{M} = \mathcal{M}^T$, $\mathcal{C} = \mathcal{C}^T$ and $\mathcal{K} = \mathcal{K}^T$. Therefore, the deduced SOFM meets the *structural conditions*.

The symmetry of $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{K}$ is ensured with no other condition but the even dimension of the original matrix A. Therefore, all single-input state-space realizations of even dimensions can be formulated in a second-order form with diagonal matrices. The matrices will have real coefficients if A is real, and will be positive definite if the original realization is stable.

The whole process is summarized in Algorithm 1. Note that the presented algorithm must solve an eigenvalue problem. Other steps are the ordering and multiplication of matrices. Therefore, this algorithm fails only if no-distinct eigenvalues appear.

Algorithm 1 *State-space Realization to a Second-Order Model (SS2SOFM).*

Input: A, B, C
Output: $\mathcal{M}, \mathcal{C}, \mathcal{K}, F, G_1, G_2$
if $B \notin \mathbb{R}^{2n_q \times 1}$ **or** $A \notin \mathbb{R}^{2n_q \times 2n_q}$ **or** $\mathrm{Re}(\lambda_i(A)) \geq 0$ **then**
  return
else
  solve $\Phi A = \Lambda \Phi$
  if $\lambda_i(A) \neq \lambda_j(A) \quad \forall \quad i \neq j$ **then**
    construct $\Lambda_1 = \mathrm{diag}(\Lambda_c, \Lambda_{r1})$,
      $\Lambda_2 = \mathrm{diag}(\bar{\Lambda}_c, \Lambda_{r2})$ (Eqn. (6))
      and associated matrices Φ_1, Φ_2 (Eqn. (7))
    compute $B_d = \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix} B$
    construct $X_1 = \mathrm{diag}(b_{d1}, \ldots, b_{dn_q})$ and
      $X_2 = \mathrm{diag}(b_{dn_q+1}, \ldots, b_{dn})$ (Eqn. (13))
    set $\mathcal{M} = \mathbb{I}$
    set $\mathcal{C} = \Lambda_1 \Lambda_2$
    set $\mathcal{K} = -(\Lambda_1 + \Lambda_2)$
    set $F = \mathbb{1}_{nq \times 1}$
    set $G_1 = C\,(\Phi_1 X_1 \Lambda_2 - \Phi_2 X_2 \Lambda_1)$
    set $G_2 = C\,(\Phi_2 \Lambda_1 - \Phi_1 \Lambda_2)$
  else
    return
  end if
end if

With Eqn. (4), an SOFM can be computed in a state-space realization. Thanks to Algorithm 1, the reverse transformation is available. Therefore, the SOFM can be reduced by reducing the associated state-space representation. The next section applies this method to reduce a model by modal truncation and balanced truncation.

4. Reduction of a single-input SOFM

A state-space realization can be reduced using two projection matrices $P \in \mathbb{R}^{2\hat{n}_q \times 2n_q}$ and $Q \in \mathbb{R}^{2n_q \times 2\hat{n}_q}$ to transform the original model into a state-space realization of a lower dimension. The projection is applied to the system as follows:

$$\hat{A} = PAQ, \quad \hat{B} = PB, \quad \hat{C} = CQ. \tag{24}$$

Among all the methods to define projection matrices, two methods are under consideration—balanced truncation and modal truncation.

4.1. Balanced truncation with the preservation of the *structural conditions*. Balanced truncation neglects the least controllable and observable states of the system based on the reachability Gramian W_r and the observability Gramian W_o. The Gramians satisfy the following two Lyapunov equations:

$$\begin{aligned} AW_r + W_r A^T + BB^T &= 0, \\ A^T W_o + W_o A + C^T C &= 0. \end{aligned} \tag{25}$$

In order to truncate the least controllable and least observable states, balanced truncation computes the transformation matrices P and Q, which balances the system, i.e., computes a model where the Gramians are equal and diagonal ($W_r = W_o = \mathrm{diag}(\sigma_i)$, where σ_i are the Hankel singular values).

To compute P and Q, let first the Cholesky decomposition be $W_r = R_c^T R_c$ and $W_o = R_o^T R_o$. Then the singular value decomposition of $R_o R_r^T = U \Sigma V^T$ computes the Hankel singular values $\Sigma = \mathrm{diag}(\sigma_i)$. Ordering U and V such that σ_i occur in decreasing order allows the truncation of the system according to the negligible Hankel singular values, i.e., the truncation of the least controllable and observable states using the following two matrices:

$$\begin{cases} Q \text{ denotes the first } 2\hat{n}_q \text{ columns of } R_r^T V \Sigma^{-\frac{1}{2}}, \\ P \text{ denotes the first } 2\hat{n}_q \text{ rows of } \Sigma^{-\frac{1}{2}} U^T R_o. \end{cases} \tag{26}$$

For more information about balanced systems and balanced truncation, see the work of Moore (1981) and Glover (1984).

If the original model is real, stable and controllable, balanced truncation ensures that the reduced state-space realization will have the same properties. Therefore, Algorithm 2 helps to balance and truncate an SOFM with the efficiency equivalent to classic state-space balanced truncation and with the preservation of the *structural conditions*.

4.2. Modal truncation. Modal truncation consists in analyzing and selecting dominant modes of the original system. Hence, the projection matrices P and Q are defined by the eigenvalues decomposition $\Phi^{-1} A \Phi$:

$$\begin{cases} Q \text{ denotes the first } 2\hat{n}_q \text{ columns of } \Phi, \\ P \text{ denotes the first } 2\hat{n}_q \text{ rows of } \Phi^{-1}. \end{cases} \tag{27}$$

Algorithm 2 *Balanced Truncation with the Preservation of the* Structural Conditions *(BTPSC).*

Input: $\mathcal{M}, \mathcal{C}, \mathcal{K}, F, G_1, G_2$
Output: $\hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}, \hat{F}, \hat{G}_1, \hat{G}_2$
compute A, B and C from Eqn. (4)
compute W_r and W_o from the Lyapunov Eqn. (25)
compute P and Q from Eqn. (26)
compute $\hat{A}$, $\hat{B}$ and $\hat{C}$ from Eqn. (24)
compute $(\hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}, \hat{F}, \hat{G}_1, \hat{G}_2) =$ SS2SOFM$(\hat{A}, \hat{B}, \hat{C})$ from Algorithm 1

For the modal truncation of a state-space realization, a rule for the truncation is currently to eliminate the eigenvalues which have the fewest real parts. For a second-order modal truncation, the same rule applies but, in addition, to preserve the even dimension, the eigenvalues are truncated by pair. If the truncated eigenvalue is complex, the conjugate eigenvalue must also be truncated. If the truncated eigenvalue is real, the next eigenvalue which has the fewest real parts must be also truncated.

According to these rules, Algorithm 3 computes a second-order modal truncation with the preservation of the *structural conditions*.

Algorithm 3 *Modal Truncation with the Preservation of the* Structural Conditions *(MTPSC).*

Input: $\mathcal{M}, \mathcal{C}, \mathcal{K}, F, G_1, G_2$
Output: $\hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}, \hat{F}, \hat{G}_1, \hat{G}_2$
compute A, B and C matrix from Eqn. (4)
solve $\Phi A = \Lambda \Phi$
for $j = 1$ to n_q **do**
  select λ_i the eigenvalue with the greatest real part
  compute $\Lambda(2j-1, 2j-1) = \lambda_i$
  if λ_i is complex **then**
    $\Lambda(2j, 2j) = \bar{\lambda}_i$
  else
    select λ_i real with the greatest real part
    compute $\Lambda(2j, 2j) = \lambda_i$
  end if
end for
compute Φ according to Λ
compute P and Q according to (27)
compute $\hat{A}$, $\hat{B}$ and $\hat{C}$ from Eqn. (24)
compute $(\hat{\mathcal{M}}, \hat{\mathcal{C}}, \hat{\mathcal{K}}, \hat{F}, \hat{G}_1, \hat{G}_2) =$ SS2SOFM$(\hat{A}, \hat{B}, \hat{C})$ from Algorithm 1

5. Numerical examples

To show the effectiveness of the proposed approach, consider two numerical examples of a single-input SOFM reduction using SLICOT benchmark models (Chahlaoui *et al.*, 2002):

- The *building model* is a model of an eight-floor building where the generalized coordinates are the displacement in the x direction, the y direction, and one rotation of each floor.
- The *clamped beam model* is a model of a clamped beam where the input is a force applied to the free end and the output is the resulting displacement.

The proposed methods are compared with the Guyan reduction (Guyan, 1964) and the Improved Reduction System (IRS) method (Friswell *et al.*, 1995) on the one hand, and with three Second-Order Balanced Truncation (SOBT) methods on the other.

In order to compare these methods, an approximation error is computed. The criterion used is the relative error between the original model and the reduced model given by

$$\frac{||\Sigma_{\text{sofm}} - \hat{\Sigma}_{\text{sofm}}||_{\mathcal{H}_\infty}}{||\Sigma_{\text{sofm}}||_{\mathcal{H}_\infty}}, \tag{28}$$

where $||\Sigma_{\text{sofm}} - \hat{\Sigma}_{\text{sofm}}||_{\mathcal{H}_\infty}$ is the $\mathcal{H}_\infty$-norm of the error model defined by the difference between the truncated and the original model and $||\Sigma_{\text{sofm}}||_{\mathcal{H}_\infty}$ is the $\mathcal{H}_\infty$-norm of the original model. The approximation errors of SOBT, SOBTp, SOBTpv come from the work of Reis and Stykel (2007).

1. The *Guyan reduction* is based on a sub-structuring partition of the undamped model (i.e., $\mathcal{C} = 0$) into two sets of complementary generalized coordinates:

$$\begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{29}$$

where vector q_1 includes the generalized coordinates which are kept and q_2 includes the generalized coordinates which are neglected. The omission of the equivalent inertia terms of the neglected coordinates (i.e., $\mathcal{M}_{21}\ddot{q}_1 + \mathcal{M}_{22}\ddot{q}_2 = 0$) in (29) gives the dependence between the kept and neglected coordinates:

$$q_2 = -\mathcal{K}_{22}^{-1}\mathcal{K}_{12} q_1. \tag{30}$$

Therefore, the reduction matrix T_g is

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} I \\ -\mathcal{K}_{22}^{-1}\mathcal{K}_{21} \end{pmatrix} q_1 = T_g q_1. \tag{31}$$

This reduction matrix is applied to the original SOFM as follows:

$$\begin{cases} \hat{\mathcal{M}} = T_g^T \mathcal{M} T_g, \\ \hat{\mathcal{C}} = T_g^T \mathcal{C} T_g, \\ \hat{\mathcal{K}} = T_g^T \mathcal{K} T_g, \end{cases} \quad \text{and} \quad \begin{cases} \hat{F} = T_g^T F, \\ \hat{G}_1 = G_1 T_g, \\ \hat{G}_2 = G_2 T_g. \end{cases} \tag{32}$$

2. The *IRS method* takes account of the inertia terms in the neglected part of the reduced model. The undamped free vibration problem of the reduced model $\hat{\mathcal{M}}\ddot{q}_1 + \hat{\mathcal{K}}q_1 = 0$ gives

$$\ddot{q}_1 = -\hat{\mathcal{M}}^{-1}\hat{\mathcal{K}}q_1. \tag{33}$$

By differentiating (30),

$$\ddot{q}_2 = -\mathcal{K}_{22}^{-1}\mathcal{K}_{21}\ddot{q}_2. \tag{34}$$

Substituting (33) and (34) in (29) gives

$$q_2 = \Big(\mathcal{K}_{22}^{-1}\left(\mathcal{M}_{21} - \mathcal{M}_{22}\mathcal{K}_{22}^{-1}\mathcal{K}_{21}\right)\hat{\mathcal{M}}^{-1}\hat{\mathcal{K}} - \mathcal{K}_{22}^{-1}\mathcal{K}_{21}\Big)q_1. \tag{35}$$

The formulation $\mathcal{K}_{22}^{-1}\left(\mathcal{M}_{21} - \mathcal{M}_{22}\mathcal{K}_{22}^{-1}\mathcal{K}_{21}\right)$ can be replaced by $S\mathcal{M}T_g$ with

$$S = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{K}_{21}^{-1} \end{pmatrix}.$$

Finally, the reduction matrix is

$$T_{irs} = T_g + S\mathcal{M}T_g\hat{\mathcal{M}}^{-1}\hat{\mathcal{K}}. \tag{36}$$

This reduction matrix is applied to the original SOFM as follows:

$$\begin{cases} \hat{\mathcal{M}} = T_{irs}^T\mathcal{M}T_{\mathrm{irs}}, \\ \hat{\mathcal{C}} = T_{irs}^T\mathcal{C}T_{\mathrm{irs}}, \\ \hat{\mathcal{K}} = T_{irs}^T\mathcal{K}T_{\mathrm{irs}}, \end{cases} \quad \text{and} \quad \begin{cases} \hat{F} = T_{\mathrm{irs}}^T F, \\ \hat{G}_1 = G_1T_{\mathrm{irs}}, \\ \hat{G}_2 = G_2T_{\mathrm{irs}}. \end{cases} \tag{37}$$

3. *SOBT reduction.* Three Second-Order Balanced Truncation (SOBT) methods are considered. These are based on the definition of a pair of second-order Gramians, called *position* and *velocity* Gramians. The first definition of second-order Gramians is given by Meyer and Sirnivasan (1996). Since the work by Sorensen and Antoulas (2004), other definitions of Gramians have been given, which are mostly used. There are different balancing techniques for second-order form models. Based on a state-space realization approach, Chahlaoui *et al.* (2006) balance both the *position* and *velocity* Gramians with an SOBT algorithm. Stykel (2006) as well as Reis and Stykel (2007) deal directly with the SOFM. According to Gramians, which are equal and diagonal, two algorithms are presented. The first one, called SOBTp, balances position Gramians, while the second one, called SOBTpv, balances the position and velocity Gramians. Note that, in order to preserve the *structural conditions* of an SOFM, SOBTpv helps to compute a symmetric second-order reduced form model if the original SOFM is symmetric. A symmetric SOFM meets the *structural conditions*, and its input matrix is the transpose of its output matrix, i.e., $G_2 = 0$ and $F = G_1^T$. In the same way, Yan *et al.* (2008) present the Second-order Balanced truncation for Passive Order Reduction (SBPOR) algorithm which preserves the *structural conditions* in the symmetric case. However, neither of these techniques of second-order balanced truncation fulfils the *structural conditions* for nonsymmetric SOFMs.

5.1. Building model. The building model has $n_q = 48$ generalized coordinates, $m = 1$ input and $p = 1$ output. The reduced model has a dimension of $n_q = 4$ generalized coordinates. The matrices computed by Algorithm 2 (BTPSC) are

$$\hat{\mathcal{M}} = \mathbb{I}, \tag{38}$$

$$\hat{\mathcal{C}} = \begin{pmatrix} 0.55 & 0 & 0 & 0 \\ 0 & 0.58 & 0 & 0 \\ 0 & 0 & 1.06 & 0 \\ 0 & 0 & 0 & 1.71 \end{pmatrix}, \tag{39}$$

$$\hat{\mathcal{K}} = \begin{pmatrix} 33.32 & 0 & 0 & 0 \\ 0 & 27.98 & 0 & 0 \\ 0 & 0 & 183.55 & 0 \\ 0 & 0 & 0 & 591.61 \end{pmatrix}, \tag{40}$$

$$\hat{F} = \mathbb{1}_{nq\times 1}, \tag{41}$$

$$\hat{G}_1 = \begin{pmatrix} -0.005 & 0.004 & -0.008 & -0.021 \end{pmatrix}, \tag{42}$$

$$\hat{G}_2 = \begin{pmatrix} 0.001 & 0.003 & 0.004 & 0.002 \end{pmatrix}. \tag{43}$$

As expected, the three matrices $\hat{\mathcal{M}}$, $\hat{\mathcal{C}}$ and $\hat{\mathcal{K}}$ are positive definite, diagonal with real entries. The input matrix $\hat{F}$ and the output matrices $\hat{G}_1$ and $\hat{G}_2$ have real entries. Because $\hat{\mathcal{M}}$, $\hat{\mathcal{C}}$ and $\hat{\mathcal{K}}$ are all diagonal, the reduced model is composed of four independent elementary oscillators where the output is a linear combination of position and velocity.

Table 1 gives the relative error for a fourth-order reduced model computed by BTPSC, MTPSC, SOBT, SOBTp, SOBTpv, Guyan and IRS. The last column indicates if the reduced model respects the *structural conditions*.

Figure 1 presents the Bode diagram of a full-order building model with the model reduced using BTPSC

Table 1. Relative errors for a fourth-order reduced model of a building.

Reduction method	Relative error	Structural conditions
BTPSC	0.144	yes
MTPSC	0.319	yes
SOBT	0.352	no
SOBTp	0.349	no
SOBTpv	0.295	no
Guyan	0.823	yes
IRS	0.757	yes

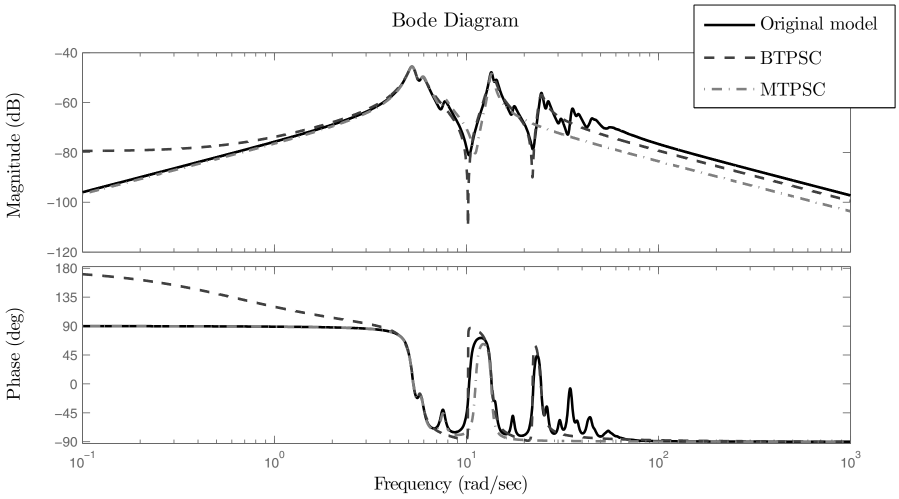

Fig. 1. Bode diagram of the full-order building model and its reduced model with BTPSC and MTPSC algorithms.

and MTPSC algorithms. In low frequencies, the MTPSC model, best approximates the original model while the BTPSC model has the best approximation in high frequencies.

5.2. Clamped beam model. The second model is a clamped beam model with $n_q = 348$ generalized coordinates, $m = 1$ input and $p = 1$ output. The reduced model has a dimension of $n_q = 17$ generalized coordinates. The relative error of the reduced models computed by BTPSC, MTPSC, SOBT, SOBTp, SOBTpv, Guyan and IRS algorithms is presented in Table 2. Again, the best relative error is given by BTPSC.

Figure 2 presents the Bode diagram of the original model, the reduced model computed using BTPSC and MTPSC. The BTPSC reduced model approximates the original model in all frequencies for the magnitude and in low frequencies for the phase. Unlike the BTPSC reduced model, the MTPSC reduced model does not approximate the original model over 1Hz in a satisfactory way.

Table 2. Relative errors for a seventeenth-order reduced model of a clamped beam.

Reduction method	Relative error	Structural conditions
BTPSC	$1.75e^{-5}$	yes
MTPSC	$1.27e^{-3}$	yes
SOBT	$1.31e^{-4}$	no
SOBTp	$1.63e^{-4}$	no
SOBTpv	$4.69e^{-4}$	no
Guyan	$9.93e^{-1}$	yes
IRS	2.12	yes

6. Conclusion

In this paper, the problem of SOFM reduction has been investigated using an equivalent state-space realization of the SOFM. To obtain a reduced model in a second-order form, a new method to transform a single-input state-space with an even dimension into a SOFM has been proposed. If the reduced model is stable, controllable, with real entries and a diagonalizable state matrix A, the SOFM meets the *structural conditions*, and hence, the reduced model is physically feasible.

This solution is suitable for single-input systems; therefore, the application field remains limited. An extension of the method to multi-input systems will be considered in further studies.

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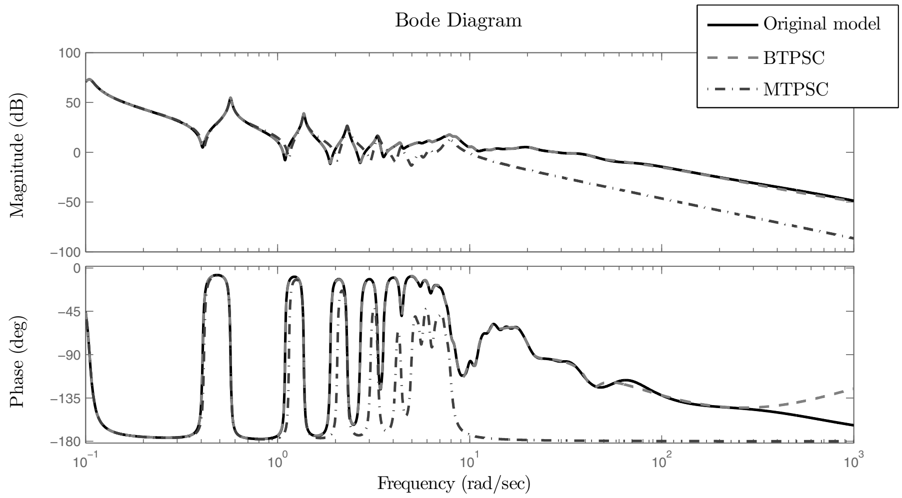

Fig. 2. Bode diagram of a full-order clamped beam model and its reduced model using BTPSC and MTPSC algorithms.

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